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Taylor–Görtler instabilities of Tollmien–Schlichting waves and other flows governed by the interactive boundary-layer equations

By PHILIP HALL AND JAMES BENNETT
Mathematics Department, North Park Road, University of Exeter, Exeter, UK

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The Taylor–Görtler vortex instability equations are formulated for steady and unsteady interacting boundary-layer flows. The effective Görtler number is shown to be a function of the wall shape in the boundary layer and the possibility of both steady and unsteady Taylor–Görtler modes exists. As an example the steady flow in a symmetrically constricted channel is considered and it is shown that unstable Görtler vortices exist before the boundary layers at the wall develop the Goldstein singularity discussed by Smith & Daniels (1981). As an example of an unsteady spatially varying basic state we also consider the instability of high-frequency large-amplitude two- and three-dimensional Tollmien–Schlichting waves in a curved channel. It is shown that they are unstable in the first 'Stokes-layer stage' of the hierarchy of nonlinear states discussed by Smith & Burggraf (1985). This instability of Tollmien–Schlichting waves in an internal flow can occur in the presence of either convex or concave curvature. Some discussion of this instability in external flows is given.

1. Introduction

Our concern is with the Taylor–Görtler instability of interactive boundary-layer flows of the type which occur in triple-deck theory. Thus, we investigate the instability of the 'lower-deck' boundary layer which is set up when a classical boundary layer with Reynolds number Re encounters a hump of height and length order $Re^{-\frac{5}{8}}$ and $Re^{-\frac{3}{8}}$ respectively. We find that the form of the Taylor–Görtler instability equations in the lower deck are almost identical with those appropriate to a classical boundary layer. The main difference is that the wall shape function $f(X,T)$ enters the instability equations and, in fact, for steady flows f_{XX} plays the role of the Görtler number.

We shall see that for unsteady interactive boundary-layer flows both steady and unsteady Taylor–Görtler vortices of the type discussed by Hall (1982, 1983) and Seminara & Hall (1976) respectively are possible. These flows are also potentially unstable to short-wavelength Rayleigh modes and the reader is referred to the papers by Smith & Bodonyi (1985) and Tutty & Cowley (1986) for a discussion of that problem. In addition Tollmien–Schlichting waves are a possible source of instability in these flows; here we concentrate on the Taylor–Görtler mechanism. In general the instability equations which we derive must be solved numerically because a parallel flow and/or a quasi-steady approximation cannot be justified. However, in order to demonstrate that some of these flows are unstable we shall here concentrate on two problems for which some asymptotic progress is possible.

First, we look in detail at the steady flow in a symmetrically constricted channel. Smith & Daniels (1981) have shown that when h the scaled height of constriction becomes large a classical boundary layer of thickness $O(h^{-\frac{1}{2}})$ is set up within the wall boundary layer. This inner boundary layer develops a Goldstein singularity beyond the minimum channel width position but Smith and Daniels showed that the singularity could be removed without any upstream influence being set up. Here we consider the instability of the flow before the boundary layer develops the singularity. For large values of h we are able to solve the instability equations asymptotically and demonstrate the instability of the $h^{-\frac{1}{2}}$ layer in the presence of concave curvature. In general the instability occurs before the Goldstein singularity develops. However, it is possible to choose humps with the required concave curvature only beyond the position where the singularity develops. In the latter case the instability mechanism of the $h^{-\frac{1}{2}}$ layer does not occur. The Smith–Daniels calculation also applies to non-symmetric channel flows and to external boundary layers; thus the instability mechanism which we describe in §3 also applies to these flows.

Secondly, we look at the unsteady interactive boundary layer which governs the growth of Tollmien–Schlichting waves in parallel or boundary-layer flows. Here the unsteadiness is characterized by Ω the frequency of the Tollmien–Schlichting wave. For definiteness, and to avoid the complications of boundary-layer growth, we look at the instability of the waves in a slightly curved channel. Recently Smith & Burggraf (1985) looked at the structure of high-frequency large-amplitude Tollmien–Schlichting waves in a variety of situations. Dependent on the size of the disturbance and the particular flow under investigation they found a hierarchy of nonlinear partial differential systems to describe the disturbance. The first nonlinear stage discussed by Smith & Burggraf is such that the disturbance has the form of a Stokes layer near the wall. We investigate the instability of this flow and identify the critical disturbance size above which the Tollmien–Schlichting wave is unstable to the Stokes layer Taylor–Görtler mode identified by Seminara & Hall (1976). When the Tollmien–Schlichting wave has amplitude greater than this critical value a three-dimensional flow containing streamwise vortices develops. The effect of this new flow on the growth of the wave into the larger amplitude states of Smith & Burggraf is not yet known. The generalization of the instability analysis of two-dimensional Tollmien–Schlichting waves to oblique waves is given in Appendix B. It was shown by Hall (1985) that a weak crossflow destroys the Görtler instability mechanism in a classical boundary layer unless the chordwise and spanwise velocity components are proportional. In the latter case the instability calculation can be reduced to an equivalent two-dimensional Görtler problem. We show that this is the case for oblique Tollmien–Schlichting waves. We can then show that a Tollmien–Schlichting wave of a given high frequency is most unstable when it is two-dimensional. In Appendix A we show that the calculation of the instabilities of Tollmien–Schlichting waves in channels is valid in other situations.

In §2 of this paper we derive the equations governing the centrifugal instability of 'lower-deck' boundary-layer flows. These equations are valid in other interactive boundary-layer flows such as the Smith–Daniels problem discussed in §3. In §4 we investigate the instability of two-dimensional Tollmien–Schlichting waves in curved channel flows whilst in §5 we give some further discussion. Finally in Appendix A and Appendix B our results of §4 are generalized to other flows.

2. The Taylor–Görtler instability equations for triple-deck flows

It is useful at this stage to discuss briefly the Taylor–Görtler instability equations for a classical two-dimensional boundary-layer flow over a curved wall. The reader is referred to the papers by, for example, Görtler (1940), Smith (1955), Floryan & Saric (1979) and Hall (1982) for a discussion of the approximations required to obtain a self-consistent set of linear stability equations.

Suppose that the l and U^* are typical length and velocity scales for the flow in the x-direction and that ν is the kinematic viscosity of the fluid. We define a Reynolds number Re by

$$Re = \frac{U^* l}{\nu}, \tag{2.1}$$

and take x and y to be dimensionless variables measuring distance along and normal to a surface with local curvature $a^{-1}K(x)$. The variable x is scaled on l whilst y is a boundary-layer variable scaled on $lRe^{-\frac{1}{2}}$. The basic flow $\boldsymbol{u}_\mathrm{B}$ is of the form

$$\boldsymbol{u}_\mathrm{B} = U^*(\overline{u}(x,y), Re^{-\frac{1}{2}}\overline{v}(x,y), 0) + \dots,$$

and this flow is perturbed by writing

$$\boldsymbol{u} = \boldsymbol{u}_\mathrm{B} + U^*(U(x,y), Re^{-\frac{1}{2}}V(x,y), Re^{-\frac{1}{2}}W(x,y)) \exp\{\mathrm{i}Re^{\frac{1}{2}}kz\}. \tag{2.2}$$

Here k is a non-dimensional wavenumber in the spanwise direction and we have assumed that the instability occurs on the boundary-layer scale.

From the momentum and continuity equations we can show that in the limit $Re \to \infty$ with the Görtler number $G = 2Re^{\frac{1}{2}}(l/a)$ held fixed the linear stability equations are

$$U_x + V_y + \mathrm{i}kW = 0, \tag{2.3a}$$

$$\overline{u}U_x + U\overline{u}_x + \overline{v}U_y + V\overline{u}_y = \{\partial_y^2 - k^2\}\, U, \tag{2.3b}$$

$$\overline{u}V_x + U\overline{v}_x + \overline{v}V_y + V\overline{v}_y + K(x)\, G\overline{u}U = -P_y + \{\partial_y^2 - k^2\}\, V, \tag{2.3c}$$

$$\overline{u}W_x + \overline{v}W_y = -\mathrm{i}kP + \{\partial_y^2 - k^2\}\, W. \tag{2.3d}$$

Here P is the non-dimensional pressure perturbation corresponding to (U, V, W) and we have assumed that the perturbation is steady. The generalization of (2.3*a–d*) to a weakly three-dimensional boundary layer is given by Hall (1985).

The essential difficulty with (2.3*a–d*) is that for G and $k = O(1)$ there is no rational reason why a parallel-flow approximation should be made and the partial differential system must be solved numerically as was done by Hall (1983). For $k \gg 1$, but $G \approx k^4$ an asymptotic solution to (2.3*a–d*) was given by Hall (1982) who showed that in this limit non-parallel effects can be taken care of in a self-consistent manner. For $O(1)$ wavenumbers the numerical calculations of Hall (1983) showed that the position of neutral stability is a function of the initial disturbance. At higher Görtler numbers the local growth rate approaches the asymptotic result which, in this regime, is consistent with a parallel-flow theory calculation. It has been assumed elsewhere that this latter result justifies the use of parallel-flow theories. However, in the only regime where the parallel-flow theories are valid, i.e. $k \gg 1$, an asymptotic result of at least the same accuracy as any parallel-flow theory can be written down in closed form.

We now show how the equations corresponding to (2.3*a–d*) can be derived for a basic flow governed by some interactive boundary-layer structure. For definiteness we focus on a flow governed by triple-deck theory. We stress that the formulation for other interactive boundary-layer structures is essentially identical. Consider then the flow over the wall

$$y = \epsilon^5 f(X, T),$$

where $X = \epsilon^{-3}x$, and $T = \epsilon^{-2}(tU^*/l)$. Here t denotes time whilst the small parameter $\epsilon = Re^{-\frac{1}{8}}$. We define the lower-deck variable Y by

$$Y = \epsilon^{-5}y,$$

and in the lower deck the basic state expands as

$$\frac{\boldsymbol{u}_{\mathrm{BL}}}{U^*} = (\epsilon\overline{u}(X, Y), \epsilon^3\overline{v}(X, Y), 0) + \dots,$$

whilst the pressure expands as

$$\frac{P}{\rho U^{*2}} = \epsilon^2\overline{p}(X) + \dots.$$

The equations which determine the flow in the lower deck are

$$\overline{u}_T + \overline{u}\overline{u}_X + \overline{v}\overline{u}_Y = -\overline{p}_X + \overline{u}_{YY}, \tag{2.4a}$$

$$\overline{u}_X + \overline{v}_Y = 0, \tag{2.4b}$$

whilst the boundary conditions at the wall are

$$\overline{u} = 0, \quad \overline{v} = f_T \quad \text{on } Y = f(X, T), \tag{2.5a}$$

and at infinity we require

$$\overline{u} \to Y + A(X, T), \tag{2.5b}$$

where A is the displacement function which must be related to $\overline{p}$ through a pressure–displacement law. If we make the unsteady Prandtl transform

$$Y \to Y + f(X, t), \quad \overline{v} \to \overline{v} + \overline{u}f_X + f_T, \tag{2.6}$$

then (2.4a), (2.4b) are unchanged whilst (2.5a, b) reduces to

$$\begin{gathered} \overline{u} = \overline{v} = 0, \quad Y = 0, \\ \overline{u} \to Y + A + f, \quad Y \to \infty. \end{gathered} \tag{2.7}$$

We now investigate the instability of this flow to Taylor–Görtler vortices which might be associated with either the steady or unsteady components of the basic state. We look for perturbations confined to the lower deck and having spanwise wavelength comparable with the lower-deck thickness. The possible source of instability is, of course, the curvature of the wall in the lower deck. We write

$$\frac{\boldsymbol{u}}{U^*} = \boldsymbol{u}_{\mathrm{BL}} + \Delta(\epsilon U(X, Y, T), \epsilon^3 V(X, Y, T), \epsilon^3 W(X, Y, T))\, E, \tag{2.8}$$

where $\Delta \ll 1$ and $E = \exp(\mathrm{i}kZ/\epsilon^5)$. Here Z has been scaled on l and we have assumed in (2.8) that the normal and spanwise velocity components are comparable. This is the usual case for Taylor–Görtler instabilities and the relative scaling of the X and Y velocity components is again consistent with that appropriate to the classical Taylor vortex problem (see, for example, Davey 1962). The pressure perturbation p^+ in the lower deck expands as

$$\frac{p^+}{\rho U^{*2}} = \epsilon^6 \Delta P(X, Y, T)\, E, \tag{2.9}$$

and the above relatively small scaling for P enables us to retain the convective and diffusive terms in the Y and Z momentum equations. It remains for us to substitute the perturbed flow into the Navier–Stokes equations for the lower deck with $\Delta \ll 1$ and linearize about the basic state. We note that at this stage it has not been necessary to define a Görtler number for the flow. After linearizing about the basic state and making the Prandtl transform (2.6) together with

$$V \rightarrow V + Uf_X,$$

we obtain

$$\left.\begin{aligned} U_X + V_Y + \mathrm{i}kW &= 0,\\ U_T + \bar{u}U_X + U\bar{u}_X + \bar{v}U_Y + V\bar{u}_Y &= \{\partial_Y^2 - k^2\}\, U,\\ V_T + \bar{u}V_X + U\bar{v}_X + \bar{v}V_Y + V\bar{v}_Y + 2U\{f_{XX}\bar{u} + f_{XT}\} &= -P_Y + \{\partial_Y^2 - k^2\}\, V,\\ W_T + \bar{u}W_X + \bar{v}W_Y &= -\mathrm{i}kP + \{\partial_Y^2 - k^2\}\, W, \end{aligned}\right\} \quad (2.10)$$

which must be solved subject to

$$U = V = W = 0, \quad Y = 0, \quad Y \rightarrow \infty. \tag{2.11}$$

We see that the generalization of (2.3) to an unsteady triple-deck flow leads to almost the same equations but with $KG\bar{u}$ replaced by $2\{f_{XX}\bar{u} + f_{XT}\}$. For steady triple-deck flows this means that f_{XX} plays the role of the Görtler number, whilst for unsteady flows an extra term proportional to Uf_{XT} arises due to the vertical motion of the boundary. For time-periodic basic states the system (2.10) contains the terms which lead to centrifugal instabilities in Stokes layers. Thus in general we must be alert to the possibility of both types of Görtler instability.

The solution of (2.10) is in general a numerical problem which, for time-dependent flows, will be an order of magnitude more difficult than the steady state calculations of Hall (1983). In the next section we will look at a particular steady state for which it is possible for us to solve (2.10) asymptotically in a self-consistent manner. As stated previously the equations (2.10) are valid for any interactive boundary-layer flow. In general the derivation of these equations follows that given above but with the following scalings to be made. First the x, y, and z disturbance velocity components scale with the x, y, and y basic state velocity components respectively. Secondly the y and z variations of the basic state are on the interaction-layer scale. Finally the pressure perturbation is reduced in size until the pressure gradient and viscous terms in the y and z momentum equations are comparable.

3. Symmetric channel flows

In general it is not possible to solve (2.10) analytically. We now show how asymptotic methods can be used for a particular steady basic flow. We refer to the internal channel flows discussed by Smith & Daniels (1981). In that problem the wall boundary-layer thickness is $\sim Re^{-\frac{1}{3}}$ and the x variations are on an $O(1)$ lengthscale. However, if the disturbance quantities are scaled as outlined above then (2.10) still apply. The basic state satisfies (2.4a, b) but with $A = 0$ in (2.7) since the channel is symmetric.

For steady flows we saw in §2 that f_{XX} plays the role of the Görtler number so that we expect the flow to become more unstable with increasing hump height. In view of the work of Hall (1982) we might then expect that an asymptotic solution of (2.10) should be possible. The Smith–Daniels problem is a suitable candidate for such an analysis because its structure for $|f| \gg 1$ is reasonably well understood. Suppose that we write

$$f(X) = hF(X), \tag{3.1a}$$

with $h \gg 1$. In this situation a classical boundary layer possibly extending to $X = -\infty$ and of thickness $\approx h^{-\frac{1}{2}}$ is attached to the hump before separation. In this layer $\overline{u}$ and $\overline{v}$ expand as

$$\overline{u} = h\overline{U} + \dots, \tag{3.1b}$$

$$\overline{v} = h^{\frac{1}{2}}\overline{V} + \dots, \tag{3.1c}$$

where $\overline{U}$ and $\overline{V}$ are functions of X and $\zeta = h^{\frac{1}{2}}Y$, and satisfy the classical boundary-layer equations with pressure gradient $-FF_X$ and $\overline{U} \rightarrow F$ when $\zeta \rightarrow \infty$. The effective Görtler number for the inner $O(h^{-\frac{1}{2}})$ boundary layer then becomes $O(h^{\frac{7}{2}})$ so that, on the basis of the work of Hall (1982), we expect that neutral modes will have $k = O(h^{\frac{7}{8}})$. The latter paper gives the essential details for the structure of a small-wavelength Görtler vortex so that in the following discussion only the essential details of the calculation will be given. We expect that the Görtler vortices with $k = O(1)$ will then have large growth rates of order $h^{\frac{7}{4}}$. It has been pointed out to the authors by Professor F. T. Smith that Tollmien–Schlichting waves in this region can have even larger amplification rates. The question of which mode dominates the instability process requires a nonlinear calculation; we do not address that problem here. The Görtler mode will be concentrated in an internal layer of depth $h^{-\frac{11}{16}}$ which is located so as to maximize the local spatial growth rate. At the location X we assume the layer is centred on $\zeta = \overline{\zeta}(X)$ and write

$$\eta = h^{\frac{3}{16}}(\zeta - \overline{\zeta}(X)). \tag{3.2}$$

The wavenumber k is then expanded as

$$k = k_0 h^{\frac{7}{8}} + k_1 h^{\frac{1}{2}} + \dots, \tag{3.3}$$

whilst we write

$$U = \{U_0(\eta, X) + h^{-\frac{3}{16}}U_1(\eta, X) + h^{-\frac{3}{8}}U_2(\eta, X) + \dots\}\, E^*, \tag{3.4}$$

together with similar expansions for $V/h^{\frac{1}{4}}$, $W/h^{\frac{1}{16}}$ and $P/h^{\frac{15}{16}}$. Here the quantity E^* is defined by

$$E^* = \exp\left[h^{\frac{3}{4}}\int^X \{\beta_0(X) + h^{-\frac{3}{8}}\beta_1(X) + \dots\}\, \mathrm{d}X\right], \tag{3.5}$$

so that $\{\beta_i(X)\}$ determine the spatial growth of the disturbance. In fact, we will concentrate on the neutral case and to the order which we proceed here it is not necessary to distinguish between the growth rates for different flow quantities. Finally, near $\overline{\zeta}$ the basic state expands as

$$\overline{u} = h\{\overline{U}_0(X) + h^{-\frac{3}{16}}\eta\overline{U}_1(X) + h^{-\frac{3}{8}}\eta^2\overline{U}_2(X) + \dots\}, \tag{3.6a}$$

$$\overline{v} = h^{\frac{1}{2}}\{\overline{V}_0(X) + h^{-\frac{3}{16}}\eta\overline{V}_1(X) + h^{-\frac{3}{8}}\eta^2\overline{V}_2(X) + \dots\}, \tag{3.6b}$$

where

$$\overline{U}_i(X) = \frac{\overline{U}^{(i)}(X, \overline{\zeta})}{i\,!}, \quad \overline{V}_i(X) = \frac{\overline{V}^{(i)}(X, \overline{\zeta})}{i\,!}. \tag{3.6c}$$

It remains for us to substitute the above expansion into (2.10) and successively equate like powers of order $h^{-\frac{3}{16}}$. The zeroth-order problem is found to be

$$(\beta_0 \overline{U}_0 + k_0^2)\, V_0 + 2F_{XX}\overline{U}_0\, U_0 = 0, \tag{3.7a}$$

$$(\beta_0 \overline{U}_0 + k_0^2)\, U_0 + V_0 \overline{U}_1 = 0, \tag{3.7b}$$

$$\mathrm{i}P_0 + k_0 W_0 = 0, \tag{3.7c}$$

$$V_0' + \mathrm{i}k_0 W_0 = 0. \tag{3.7d}$$

The required consistency condition for (3.7*a*), (3.7*b*) yields the zeroth-order eigenrelation

$$(\beta_0 \overline{U}_0 + k_0^2)^2 = 2\overline{U}_1 \overline{U}_0 F_{XX}, \tag{3.8}$$

and the potentially unstable root of this equation is

$$\overline{U}_0 \beta_0 = -k_0^2 + (2\overline{U}_1 \overline{U}_0 F_{XX})^{\frac{1}{2}},$$

which occurs only for $\overline{U}_1 \overline{U}_0 F_{XX} > 0$. At a position where $\overline{U}_1 \overline{U}_0 F_{XX}$ vanishes we have a coalescence of modes and a transition region is required. The most unstable position in the layer is such that

$$\beta_0 \overline{U}_0(\beta_0 \overline{U}_0 + k_0^2) = [\overline{U}_1^2 + 2\overline{U}_0 \overline{U}_2] F_{XX}, \tag{3.9}$$

and, with F, k_0 given, (3.8), (3.9) fix $\overline{\zeta}$ and β_0. From now on we restrict our attention to the neutral case and set $\beta_0 = \beta_1 = \beta_2 = 0$ so that the flow is neutral at X if

$$k_0^4 = 2\overline{U}_1 \overline{U}_0 F_{XX}, \tag{3.10}$$

and $\overline{\zeta}$ is determined by the condition

$$\overline{U}_1^2 + 2\overline{U}_0 \overline{U}_2 = 0,$$

which requires that $|\overline{UU'}|$ has a maximum at $\zeta = \overline{\zeta}$. At second-order the following equation for V_0 emerges as a solvability condition:

$$V_0'' - \tfrac{4}{3}k_0 k_1 V_0 + k_0^{-2}\eta^2 F_{XX}[\overline{U}_1 \overline{U}_2 + \overline{U}_3 \overline{U}_0] V_0 = 0. \tag{3.11}$$

The solutions of (3.11) which decay when $|\eta| \to \infty$ are

$$V_0 = V_{0n} = U(-n-\tfrac{1}{2}, \gamma\eta), \tag{3.12}$$

where $U(-n-\frac{1}{2}, \gamma\eta)$ is a parabolic cylinder function and

$$\gamma = \sqrt{2}\{-F_{XX}[\overline{U}_1 \overline{U}_2 + \overline{U}_3 \overline{U}_0] k_0^{-2}\}^{\frac{1}{4}},$$

the wavenumber k, must then satisfy

$$k_1 = k_{1n} = \frac{3\gamma^2}{4k_0}\{n + \tfrac{1}{2}\}. \tag{3.13}$$

The most dangerous mode corresponds to $n = 0$ so that correct to order $h^{\frac{1}{2}}$ the neutral wavenumber is

$$k = \{2\overline{U}_1 \overline{U}_0 F_{XX}\}^{\frac{1}{4}} h^{\frac{7}{8}} + \frac{3}{4}\left\{-\frac{[\overline{U}_1 \overline{U}_2 + \overline{U}_3 \overline{U}_0]}{2\overline{U}_0 \overline{U}_1}\right\}^{\frac{1}{2}} h^{\frac{1}{2}} + \dots. \tag{3.14}$$

It follows from (3.14) that instability can occur only if $\overline{U}_1 \overline{U}_0 F_{XX}$ is positive somewhere in the flowfield. If we restrict our attention to humps with $f > 0$ then until the Goldstein singularity develops $\overline{U}_1 \overline{U}_0$ is positive. Thus such flows are potentially unstable only where the wall is locally concave. There are clearly many flows of this type which can therefore never support Taylor–Görtler instabilities; the example investigated numerically by Smith & Daniels (1981) is one.

The neutral state defined by (3.14) defines the location before or beyond which spatially growing Taylor–Görtler vortices can exist. We expect that vortices with $O(1)$ wavelengths will also be unstable and will have growth rates of order $h^{\frac{7}{4}}$ if $h \gg 1$. At $O(1)$ values of h we again expect unstable Taylor–Görtler vortices with $O(1)$ wavelengths. These speculative remarks must in due course be checked by solving (2.10) numerically.

If we assume that $F(X)$ and $k \gg 1$ are given then an alternative interpretation of (3.14) is available. In this situation we can think of (3.14) as an implicit equation for the hump height $h \gg 1$ which makes the Görtler vortex flow with wavenumber $k \gg 1$ neutral at some position in the flow. Suppose next that F_{XX} is positive in $(-\infty, -C)$ and negative in $(-C, D)$ where C and D are positive constants. Since we also know that

$$\overline{U} = 0, \quad \zeta = 0, \quad \overline{U}_\zeta \to 0, \quad \zeta \to \infty,$$

and

$$\overline{U} \approx Y + O(f), \quad X \to -\infty,$$

it follows that $\overline{U}_0 \overline{U}_1 F_{XX}$ has at least one maximum in $-\infty < X < -C, 0 < \zeta < \infty$. Suppose that the largest maximum occurs at

$$X = X_C, \quad \bar{\zeta} = \zeta_C,$$

then ignoring the order $h^{\frac{1}{2}}$ term in (3.14) we see that, for $k \gg 1$, the minimum hump height h_C which leads to a neutral vortex anywhere (in fact at (X_C, ζ_C)) is given by

$$h_C = \{2\overline{U}_1(X_C)\,\overline{U}_0(X_C)\,F_{XX}\}^{-\frac{2}{7}} k^{\frac{8}{7}}. \tag{3.15}$$

If h is increased beyond h_C there will be two neutral locations at $-X_C - \alpha$, $-X_C + \beta$ with $\alpha, \beta > 0$ each corresponding to the wavenumber k. Between these positions the flow is formally unstable, the instability will amplify by an amount $O(\exp[h^{\frac{3}{4}} I])$ for some $I > 0$ in this interval and will become nonlinear if its initial size is sufficiently large. As stated above for humps with $h = O(1)$ we expect that a similar situation arises for $k = O(1)$ with at least one finite interval where instability occurs. Beyond the position of the maximum constriction a further region where instability might occur will exist so long as the Goldstein singularity is not encountered.

4. The instability of Tollmien–Schlichting waves in curved channel flows

In the previous section we described how a particular steady solution of (2.4a, b), (2.7) becomes unstable to steady Taylor–Görtler vortices. We now show how a time-periodic solution of that system can also become unstable to a Stokes layer Taylor–Görtler vortex. The particular type of time-periodic basic state which we consider corresponds to a large-amplitude high-frequency Tollmien–Schlichting wave propagating in a curved channel. Here the curvature which causes the instability is not on the triple-deck lengthscale. Hence it is necessary to say a few words about the derivation of the appropriate form of the equations corresponding to (2.10).

We consider the flow driven by an azimuthal pressure gradient between concentric cylinders of radii a, $a+d$. The maximum flow velocity is taken to be $\frac{1}{4}U_0$ and if we define dimensionless variables x and y by

$$x = \frac{a\theta}{d} = \delta^{-1}\theta, \quad y = \frac{r-a}{d},$$

then in the absence of either Tollmien–Schlichting waves or Taylor–Görtler vortices the basic state for $\delta \ll 1$ is

$$\boldsymbol{u}_\mathrm{B} = U_0(u_0 + O(\delta), 0, 0) \tag{4.1}$$

with

$$u_0 = y(1-y). \tag{4.2}$$

The Reynolds number Re is defined by

$$Re = \frac{U_0 d}{\nu},$$

and Dean (1928) showed that (4.1) is centrifugally unstable for $Re^2\delta = O(1)$. The flow is also unstable to Tollmien–Schlichting waves for $\frac{1}{8}Re > 5774$. We ignore the steady

Taylor–Görtler mode and examine the instability of finite amplitude Tollmien–Schlichting waves to Stokes-layer Taylor–Görtler vortex modes. These occur for $Re^{\frac{4}{7}}\delta = O(1)$ and so are apparently less important than the steady vortex mode. However, we shall see that they occur both near the inner and outer cylinders so that in more general flows over convex walls we can expect this mode to be the only centrifugal one available. Our choice of the curved-channel-flow problem rather than an external boundary layer enables us to investigate the instability mechanism without the difficulties associated with the effect of boundary-layer growth. In Appendix B we discuss the external-flow problem and determine the structures which lead to the instability mechanism discussed in this section.

We now describe the large-amplitude high-frequency Tollmien–Schlichting disturbances to (4.1) which exist for $Re \gg 1$. Later we shall make a further assumption that the frequency of these disturbances is large. The description of the disturbance in this region is very similar to that of Smith & Burggraf. The latter calculation is itself related to that of Stephanoff *et al.* (1983). We take z and t to be dimensionless axial and time variables scaled on d and d/U_0. The appropriate lengthscale for a Tollmien–Schlichting wave in a channel is $O(Re^{\frac{1}{7}})$ so that, following Smith (1979), we define

$$\epsilon = Re^{-\frac{1}{7}} \tag{4.3}$$

and

$$X = \epsilon x, \tag{4.4}$$

then the wall layers at $y = 0, 1$ are of thickness $O(\epsilon^2)$. Near $y = 0$ we write

$$Y = \frac{y}{\epsilon^2}. \tag{4.5}$$

Following Smith & Burggraf we seek a flow for $Y = O(1)$ of the form

$$\boldsymbol{u} = U_0(\epsilon^2\overline{u}(X, Y, T), \quad \epsilon^5\overline{v}(X, Y, T), 0) + \ldots, \tag{4.6}$$

where $T = \epsilon^3 t$. The corresponding pressure perturbation is $\rho U_0^2 \epsilon^4 \overline{p}(X, T)$ and $\overline{u}$, $\overline{v}$, $\overline{p}$ are determined by

$$\left.\begin{aligned} \overline{u}_T + \overline{u}\overline{u}_X + \overline{v}\overline{u}_Y &= -\overline{p}_X + \overline{u}_{YY}, \\ \overline{u}_X + \overline{v}_Y &= 0, \\ \overline{u} = \overline{v} = 0, \quad & Y = 0, \\ \overline{u} \to Y + A(X,T), \quad & Y \to \infty, \end{aligned}\right\} \tag{4.7}$$

Near $y = 1$ a similar boundary layer exists whilst in the core we have

$$\left.\begin{aligned} \boldsymbol{u} &= U_0(u_0, 0, 0) + U_0(\epsilon^2 \tilde{u}(X, y, T), \quad \epsilon^3 \tilde{v}(X, y, T), 0) + \ldots, \\ p &= \rho U_0^2 \epsilon^4 \tilde{p}(X, y, T) + \ldots. \end{aligned}\right\} \tag{4.8}$$

Here $\tilde{u}$, $\tilde{v}$, $\tilde{p}$ satisfy

$$u_0 \tilde{u}_X + \tilde{v} u_{0y} = 0, \quad \tilde{u}_X + \tilde{v}_y = 0, \quad u_0 \tilde{v}_X = -\tilde{p}', \tag{4.9}$$

and the appropriate solution is

$$(\tilde{u}, \tilde{v}) = (A(X,T)\, u_{0y}, \quad -A_X(X,T)\, u_0), \tag{4.10}$$

$$\tilde{p} = A_{XX} \int^y u_0^2(s)\, \mathrm{d}s, \tag{4.11}$$

so that the required pressure–displacement relationship is

$$p|_{y=1} - p|_{y=0} = \tfrac{1}{30} A_{XX}. \tag{4.12}$$

We are interested in the large-amplitude high-frequency solutions of (4.7) discussed by Smith & Burggraf (1985). The latter authors investigated a hierarchy of high-frequency large-amplitude states beginning with the case $\partial/\partial T = O(\Omega) \gg 1$, $\overline{u} = O(1)$. We shall concern ourselves here only with the latter state and the reader is referred to the Smith–Burggraf paper for a discussion of the remarkable range of more nonlinear states which occur for $\overline{u} \gg 1$. However, our expansion differs slightly from that of Smith & Burggraf because the lower branch eigenrelations for channel flow do not coincide with those appropriate to Blasius flow. Thus for a disturbance with wavenumber α and frequency ω the eigenrelation is

$$A_i'(\xi_0) = \frac{e^{\frac{1}{6}i\pi}}{60}\alpha^{\frac{7}{3}}\int_{\xi_0}^{\infty} A_i(\zeta)\,d\zeta \tag{4.13a}$$

with
$$\xi_0 = -\omega e^{\frac{1}{6}i\pi}\alpha^{-\frac{2}{3}}. \tag{4.13b}$$

If we let α be real and large then (4.13) yields

$$\omega = \frac{\alpha^3}{60} + O(\alpha^{-\frac{1}{2}}), \tag{4.13c}$$

where the term $O(\alpha^{-\frac{1}{2}})$ has a positive imaginary part. (Note that the corresponding equation in Smith & Burggraf has $\omega \approx \alpha^2$.)

We shall first consider the instability of (4.6) to a Taylor–Görtler vortex perturbation with axial wavelength $O(\epsilon^2)$. We thus write

$$u = U_0(\epsilon^2\overline{u} + \epsilon^2[U(X, Y, T)\exp(ikz\epsilon^{-2}) + \text{c.c.}] + \ldots), \tag{4.14a}$$

$$v = U_0(\epsilon^5\overline{v} + \epsilon^5[V(X, Y, T)\exp(ikz\epsilon^{-2}) + \text{c.c.}] + \ldots), \tag{4.14b}$$

$$w = U_0(\epsilon^5[W(X, Y, T)\exp(ikz\epsilon^{-2}) + \text{c.c.}] + \ldots), \tag{4.14c}$$

$$p = \rho U_0^2(\epsilon^4\overline{p} + \epsilon^{10}[P(X, Y, T)\exp(ikz\epsilon^{-2}) + \text{c.c.}] + \ldots), \tag{4.14d}$$

and then
$$\frac{2a}{d} = 2\delta = D\epsilon^4, \tag{4.14e}$$

where D can be interpreted as a Taylor–Görtler number for the Tollmien–Schlichting wave. The equations (4.14*a–d*) are then substituted into the momentum and continuity equations and, after linearizing about the basic state, we find that (U, V, W, P) satisfies (2.10) but with the term $2U\{f_{XX}\overline{u} + f_{XT}\}$ replaced by $D\overline{u}U$. At this stage the eigenvalue problem $D = D(k)$ could in principle be solved numerically for any given basic state. We shall proceed by looking at the high-frequency limit of the Tollmien–Schlichting waves in order to make some analytical progress. We stress that we expect instability to occur for $O(1)$ frequencies but do not pursue the necessary large-scale computational task required to verify this speculative remark.

We now describe the solution of the modified form of (2.10) discussed above in the high-frequency limit. We first note that, as shown by Smith & Burggraf, in this limit the disturbance develops into a Stokes layer at the wall $Y = 0$.

We now write
$$\omega = \alpha^3\Omega = \frac{a^3}{60},$$

and define a Stokes layer variable η, timescales $\overline{T}$, $\tilde{T}$ and a fast spatial variable $\overline{X}$ by
$$\eta = \alpha^{\frac{3}{2}}Y, \quad \overline{T} = \alpha^3 T, \quad \tilde{T} = \alpha^{-\frac{1}{2}}T, \quad \overline{X} = \alpha X. \tag{4.15}$$

In the Stokes layer $\overline{u}$, $\overline{v}$, and $\overline{p}$ expand as

$$\left.\begin{aligned} \overline{u} &= \alpha^{\frac{1}{4}}[u_{01}E+\text{c.c.}]+\ldots = \alpha^{\frac{1}{4}}\overline{u}_{01}+\ldots, \\ \overline{v} &= \alpha^{-\frac{1}{4}}[v_{01}E+\text{c.c.}]+\ldots = \alpha^{-\frac{1}{4}}\overline{v}_{01}+\ldots, \\ \overline{p} &= \alpha^{\frac{9}{4}}[p_{01}E+\text{c.c.}]+\ldots = \alpha^{\frac{9}{4}}\overline{p}_{01}+\ldots, \end{aligned}\right\} \tag{4.16}$$

where

$$E = \exp \mathrm{i}[\overline{X}-\Omega\overline{T}]$$

and

$$\left.\begin{aligned} u_{01} &= \frac{p_{01}}{\Omega}[1-\mathrm{e}^{m\Omega^{\frac{1}{2}}\eta}], \\ v_{01} &= \frac{-\mathrm{i}p_{01}}{\Omega}\left[\eta-\frac{-\mathrm{e}^{m\Omega^{\frac{1}{2}}\eta}}{m\Omega^{\frac{1}{2}}}+\frac{1}{m\Omega^{\frac{1}{2}}}\right], \\ p_{01} &= p_{01}(\tilde{T}), \quad m = \exp\{\tfrac{3}{4}\mathrm{i}\pi\}. \end{aligned}\right\} \tag{4.17}$$

In fact p_{01} depends on a succession of slow timescales other than $\tilde{T}$. The corresponding expansion in Smith & Burggraf depends on a sequence of slow spatial variables. Stewart & Smith (private communication) have derived the corresponding solution for channel flows in terms of ω as the large parameter. In that case p_{01} depends on a sequence of slow spatial variables. At higher order it is found that p_{01} satisfies a Stuart–Landau equation with the cubic term having a purely imaginary coefficient. This means that p_{01} grows exponentially on the $\tilde{T}$ timescale. When this growth has reached a critical size the more nonlinear states found by Smith & Burggraf apply. Here we look at the stability of the flow whilst it is evolving on the $\tilde{T}$ timescale.

We recall that the stability of the ϵ^2 layer is governed by (2.10) with $2U\{f_{XX}\,\overline{u}+f_{XT}\}$ replaced by $D\overline{u}U$ where D is the Dean number. We are now interested in the form of those equations in the Stokes layer. We first write

$$k = k_0\alpha^{\frac{3}{2}}+\ldots, \quad D = D_0\alpha^4+\ldots, \tag{4.18a, b}$$

and then expand U, V, W, P in the form

$$U = [U_0(\overline{X},\eta,\overline{T})+O(\alpha^{-\frac{3}{2}})]\exp\left\{\int^{\tilde{T}}\alpha^{\frac{7}{2}}\sigma(\tilde{T})\,\mathrm{d}\tilde{T}\right\}, \tag{4.19a}$$

$$V = \alpha^{\frac{5}{4}}[V_0(\overline{X},\eta,\overline{T})+O(\alpha^{-\frac{3}{2}})]\exp\left\{\int^{\tilde{T}}\alpha^{\frac{7}{2}}\sigma(\tilde{T})\,\mathrm{d}\tilde{T}\right\}, \tag{4.19b}$$

$$W = \alpha^{\frac{5}{4}}[W_0(\overline{X},\eta,\overline{T})+O(\alpha^{-\frac{3}{2}})]\exp\left\{\int^{\tilde{T}}\alpha^{\frac{7}{2}}\sigma(\tilde{T})\,\mathrm{d}\tilde{T}\right\}, \tag{4.19c}$$

$$P = \alpha^{\frac{11}{4}}[P_0(\overline{X},\eta,\overline{T})+O(\alpha^{-\frac{3}{2}})]\exp\left\{\int^{\tilde{T}}\alpha^{\frac{7}{2}}\sigma(\tilde{T})\,\mathrm{d}\tilde{T}\right\}. \tag{4.19d}$$

The $\overline{X}$ variation enters the zeroth-order problem only through the $\overline{X}$ dependence of $\overline{u}_{01}$. The growth rate $\sigma(\tilde{T})$ depends on $\tilde{T}$ through the $\tilde{T}$ dependence of $\overline{p}_{01}$.

If the expansions (4.18a, b), (4.19a–d) are substituted into (2.10) the zeroth-order problem is found to be

$$V_{0\eta}+\mathrm{i}k_0W_0 = 0, \tag{4.20a}$$

$$\sigma U_0+U_{0\overline{T}}+V_0\overline{u}_{01\eta} = \{\partial_\eta^2-k_0^2\}U_0, \tag{4.20b}$$

$$\sigma V_0+V_{0\overline{T}}-D_0\overline{u}_{01}U_0 = -P_{0\eta}+\{\partial_\eta^2-k_0^2\}V_0, \tag{4.20c}$$

$$\sigma W_0+W_{0\overline{T}} = -\mathrm{i}k_0P_0+\{\partial_\eta^2-k_0^2\}W_0, \tag{4.20d}$$

which must be solved subject to

$$U_0 = V_0 = W_0 = 0, \quad \eta = 0, \tag{4.21a}$$

$$U_0,\ V_0,\ W_0 \to 0, \qquad \eta\to\infty, \tag{4.21b}$$

so that the vortex structure is confined to the Stokes layer. The eigenvalue problem specified (4.20a–d), (4.21a,b) is identical to that studied by Hall (1984) in the context of Schlichting's (1932) transversely oscillating cylinder problem. In fact, we identify (k, T) of (2.10) of Hall (1984) with $((2/\Omega)^{\frac{1}{2}}k_0, \sqrt{2}D_0|p_{01}|^2/\Omega^{\frac{7}{2}})$.

It follows from the numerical calculations of Hall (1984) for the neutral case $\sigma = 0$ that the Tollmien–Schlichting wave is formally unstable for

$$D_0|P_{01}|^2 > 8.48\Omega^{\frac{7}{2}} = 0.0000051, \tag{4.22}$$

which, for a given value of D_0, determines P_{01} the critical Tollmien–Schlichting wave amplitude.

A similar analysis governs the instability problem for the Tollmien–Schlichting wall layer at $y = 1$. The only change is that, because the layer is now on a concave wall, the sign of D in (4.20c) must be switched. Papageorgiou (1986) has investigated that eigenvalue problem and the critical state is then determined by

$$D|p_{01}|^2 > 0.000003. \tag{4.23}$$

Thus the layer at $y = 1$ becomes unstable first, and presumably the flow becomes three-dimensional before the larger amplitude two-dimensional states of Smith & Burggraf develop. It is known from the experimental work of Seminara & Hall (1976) and Park, Barenghi & Donnelly (1980) that the initial Stokes layer instability is followed by a secondary mode of instability at about 30% above the first critical Taylor number. In this regime the vortices interact and the disturbance persists beyond the Stokes layer. No adequate theoretical description of this non-equilibrium state is yet available but the consequences for the problem discussed here are important. We refer to the fact that if the Tollmien–Schlichting wave also undergoes this secondary mode of instability then there will be a mechanism for disturbances inside the Stokes layer to penetrate outside the boundary layer.

We further note that if the mechanism described here does indeed apply to external flows then Tollmien–Schlichting waves will generate Taylor–Görtler vortices if convex or concave regions exist. In the concave regions the Taylor–Görtler mechanisms associated with the main deck basic state will be more unstable so that the Tollmien–Schlichting breakdown into the Stokes layer mode is probably only of practical importance in convex or flat regions. This problem is discussed in Appendix A. In Appendix B we extend the analysis of this section to cover oblique Tollmien–Schlichting waves. We will see that two-dimensional waves are the least stable.

Finally, we note from the calculations of Hall (1984) that when instability occurs V_0, W_0, P_0 are of the form

$$\begin{Bmatrix} V_0 \\ W_0 \\ P_0 \end{Bmatrix} = \sum_{n=-\infty}^{\infty} \begin{Bmatrix} V_{0n} \\ W_{0n} \\ P_{0n} \end{Bmatrix} E^{2n+1},$$

whilst

$$U_0 = \sum_{\infty}^{\infty} U_{0n} E^{2n}.$$

Some of the functions $U_{0n}(\eta)$, $V_{0n}(\eta)$ can be found in the paper by Hall (1984).

5. Conclusion

We have shown that the interactive boundary-layer flows of the type which arise in triple-deck theory can support Taylor–Görtler vortices. The form of the equations for steady flows is identical to that found for classical boundary layers if we interpret the wall curvature as the Görtler number. For unsteady boundary layers an extra term proportional to the streamwise gradient of the wall velocity is introduced into the equations.

We have seen that Görtler instabilities of both steady and unsteady boundary layers can be described within the above framework. In particular, we showed that a large amplitude high frequency Tollmien–Schlichting wave can interact with a curved wall to give a Stokes layer Görtler vortex. Thus a two-dimensional Tollmien–Schlichting wave can develop a three-dimensional sublayer if the amplitude is sufficiently large.

In Appendix A we determine the conditions under which the analysis of §4 applies to a Tollmien–Schlichting wave in a more general basic flow. We show that the analysis applies for flows which support Tollmien–Schlichting waves with wave-numbers large compared to $Re^{-\frac{2}{7}}$. Thus for a classical subsonic boundary layer of thickness $Re^{-\frac{1}{2}}$ the analysis does not apply. However this does not mean that the instability mechanism of §4 is not operational in such flows.

In Appendix B we extend the analysis of §4 to investigate the instability of oblique Tollmien–Schlichting waves. In particular we investigate the question of which type of Tollmien–Schlichting wave is the most susceptible to the Stokes layer instability. In order to answer the question we fix the frequency of the given Tollmien–Schlichting wave and find the orientation of the wave which leads to instability at the smallest amplitude. We find that the instability of an oblique Tollmien–Schlichting wave can be related to that of an equivalent two-dimensional one. Moreover the two-dimensional Tollmien–Schlichting wave is shown to be unstable at the smallest amplitude.

It is also worth pointing out that the larger amplitude states of Smith & Burggraf are also almost certainly centrifugally unstable. The instability properties of this regime cannot be obtained so easily because the basic state in the sublayer now satisfies a nonlinear problem which must be solved numerically. We would anticipate that, with $\delta \approx Re^{-\frac{4}{7}}$, the growth rates of these states would be larger than those associated with §4. Alternatively, we would expect that these states could be centrifugally stable at smaller values of the curvature.

The main purpose of this paper has been to demonstrate asymptotically that two examples of flows governed by interactive boundary-layer theory are centrifugally unstable. We should not overlook the fact that these flows are also often unstable to possibly more unstable disturbances such as Rayleigh modes or Tollmien–Schlichting waves. However, the question of which mode will dominate the instability characteristics for a given flow cannot be answered by the type of linear analysis given here. Thus it is pointless to speculate which type of instability will be the most physically important.

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Appendix A. The instability of two-dimensional Tollmien–Schlichting waves in other flows

Here we derive the conditions which enable the instability mechanism of §4 to be operational in more general high-Reynolds-number flows. Suppose that there is a high-Reynolds-number flow past a wall of typical radius of curvature a. If d and U_0^* denote a typical length and a typical velocity along the wall we define Re and δ by

$$Re = \frac{U_0^* d}{\nu}, \quad \delta = \frac{d}{a}, \tag{A 1}$$

and it is assumed that

$$Re \gg 1, \quad \delta \ll 1. \tag{A 2}$$

If this basic state is to support Tollmien–Schlichting waves having a triple-deck structure with depth Y and length X in the normal and downstream directions we require that

$$Y \approx Re^{-\frac{1}{3}}\lambda^{-\frac{1}{3}}X^{\frac{1}{3}}, \quad \text{for any inviscid–viscous interaction} \tag{A 3}$$

and

$$\lambda^5 X^4 \approx Re \quad \text{for a triple deck.} \tag{A 4}$$

Here λ denotes the dimensionless shear stress of the basic state at the wall.

If the lower deck is to support a Görtler vortex instability the continuity and normal momentum equations show that u and v, the x and y disturbance velocity components, must satisfy

$$u_x \approx v_y \tag{A 5}$$

and

$$u\delta \approx v_x. \tag{A 6}$$

In view of (A 3) we therefore require that

$$\delta \approx YX^{-2} \approx Re^{-\frac{1}{3}}\lambda^{-\frac{1}{3}}X^{-\frac{5}{3}}. \tag{A 7}$$

Thus for the channel-flow problem of §4 we have $X \approx Re^{\frac{1}{7}}$, $\lambda = O(1)$ and then (A 7) gives

$$\delta \approx R^{-\frac{4}{7}},$$

which is the scaling used in §4. However if we wish to apply the analysis of §4 to a Tollmien–Schlichting wave governed by triple-deck theory then (A 4), (A 7) yield

$$\delta \approx Re^{-\frac{3}{4}}\lambda^{\frac{7}{4}}. \tag{A 8}$$

The original constraint $\delta \ll 1$ leads to the condition

$$\lambda^{-1} \gg Re^{\frac{3}{7}}, \tag{A 9}$$

λ^{-1} is in effect the thickness of the oncoming boundary layer and will depend on the flow under consideration. For example, in a subsonic developing boundary layer on a plate, the mechanism of §4 will only apply at distances greater than $O(Re^{\frac{1}{7}})$ from the leading edge.

Appendix B. The instability of oblique waves in channel flows

Here we briefly show how the results of §4 can be generalized to investigate the Taylor–Görtler instability of oblique Tollmien–Schlichting disturbances in channel flows. The basic state is now three-dimensional and the results of Hall (1985) for the Görtler instability of a classical three-dimensional boundary layer are relevant. It was shown in that paper that a weak crossflow applied to a two-dimensional boundary layer will in general be sufficient to prevent the occurrence of Görtler vortices. The only exception is for flows where the spanwise and chordwise velocity components are proportional. In that case the instability of the flow can be described in terms of an equivalent stability calculation for a two-dimensional flow.

It turns out that the exception described above applies to the instability problem for oblique waves in a curved channel. This is because, as shown by Smith (1986), in the high-frequency limit the spanwise and chordwise velocity components are proportional. We first note that the eigenrelation corresponding to an oblique wave with wavenumbers α and β in the x- and z-directions and frequency ω is

$$A'(\xi_0) = \frac{e^{\frac{1}{6}i\pi}}{60}\alpha^{\frac{7}{3}}\left\{1+\frac{\beta^2}{\alpha^2}\right\}\int_{\xi_0}^{\omega} A(\zeta)\,\mathrm{d} \tag{B 1a}$$

with

$$\xi_0 = -\omega e^{\frac{1}{6}i\pi}\alpha^{-\frac{2}{3}}. \tag{B 1b}$$

This is the generalization of the two-dimensional result (4.13) and for large ω yields

$$\omega \approx \frac{\alpha^3}{60}\left\{1+\frac{\beta^2}{\alpha^2}\right\}. \tag{B 2}$$

We confine our investigation to the limit $\alpha \to \infty$ with

$$\beta = \alpha B, \quad \omega = \alpha^3\Omega = \alpha^3\frac{[1+B^2]}{60},$$

where B and Ω are $O(1)$. For oblique waves the Tollmien–Schlichting wave in the lower deck develops a double-layer structure with an inner Stokes layer of thickness $\alpha^{-\frac{3}{2}}$ and an outer layer of thickness α^2. Here we study the instability of the $\alpha^{-\frac{3}{2}}$ layer. In this layer the oblique wave can be written as

$$u = \epsilon^2\{\alpha^{\frac{1}{4}}\tilde{u}_{01}\,E+\mathrm{c.c.}\}+ \;= \epsilon^2\{\alpha^{\frac{1}{4}}\overline{u}_{01}+\ldots\}, \tag{B 3a}$$

$$v = \epsilon^2\{\alpha^{-\frac{1}{4}}\tilde{v}_{01}\,E+\mathrm{c.c.}\}+ \;= \epsilon^2\{\alpha^{-\frac{1}{4}}\overline{v}_{01}+\ldots\}, \tag{B 3b}$$

$$w = \epsilon^2\{\alpha^{\frac{1}{4}}\tilde{w}_{01}\,E+\mathrm{c.c.}\}+ \;= \epsilon^2\{\alpha^{\frac{1}{4}}\overline{w}_{01}+\ldots\}, \tag{B 3c}$$

$$p = \epsilon^4\{\alpha^{\frac{9}{4}}\tilde{p}_{01}\,E+\mathrm{c.c.}\}+ \;= \epsilon^4\{\alpha^{\frac{9}{4}}\overline{p}_{01}+\ldots\}. \tag{B 3d}$$

where

$$\tilde{u}_{01} = \frac{p_{01}}{\Omega}\{1-e^{m\Omega^{\frac{1}{2}}\tilde{Y}}\},$$

$$\tilde{v}_{01} = -i(1+B^2)\frac{p_{01}}{\Omega}\left\{\tilde{Y}-\frac{e^{m\Omega^{\frac{1}{2}}\tilde{Y}}}{m\Omega^{\frac{1}{2}}}+\frac{1}{m\Omega^{\frac{1}{2}}}\right\},$$

$$\tilde{w}_{01} = Bu_{01},$$

$$p_{01} = \text{constant}, \quad m = e^{\frac{3}{4}i\pi}, \quad \tilde{Y} = \frac{\alpha^{\frac{3}{2}}y}{\epsilon^2},$$

$$\tilde{E} = \exp\{i[X+BZ-\Omega\overline{T}]\},$$

$$X = \alpha\epsilon x, \quad Z = \alpha\epsilon z, \quad \overline{T} = \alpha^3\epsilon^3 t.$$

In the outer $O(\alpha^2)$ layer the above flow adjusts to that necessary to match with the flow in the core. The amplitude p_{01} again grows exponentially on a slow timescale, for convenience we ignore that time-variation in the following instability calculation.

The flow in the inner $O(\alpha^{-\frac{3}{2}})$ layer, given by (B 3*a–d*), is essentially the same as the flow in the two-dimensional case, (4.16) and (4.17), but at an angle to the curvature.

We now perturb this flow by a Görtler vortex aligned with the velocity:

$$\begin{aligned} \boldsymbol{u}' = (&\epsilon^2(U_0\cos\phi+\ldots)+\epsilon^5(-\alpha^{\frac{5}{4}}W_0\sin\phi+\ldots), \epsilon^5(\alpha^{\frac{5}{4}}V_0+\ldots)+\ldots, \\ &\epsilon^2(U_0\sin\phi+\ldots)+\epsilon^5(\alpha^{\frac{5}{4}}W_0\cos\phi+\ldots))\,E'+\text{c.c.}, \quad \text{(B 4)} \\ P' &= \epsilon^{10}\alpha^{\frac{11}{4}}P_0 E'+\text{c.c.} \end{aligned}$$

where $$E' = \exp\left(\frac{\mathrm{i}k_0\alpha^{\frac{1}{2}}}{\epsilon^3}(-X\sin\phi+Z\cos\phi)-\mathrm{i}\overline{T}\right),$$

and U_0, V_0, W_0 and P_0 are functions of ξ, $\tilde{Y}$ and $\overline{T}$,

where $$\xi = X\cos\phi+Z\sin\phi.$$

Here ϕ is the angle that the basic flow (B $3a$–d) makes with the curvature and is given by

$$\tan\phi = B. \tag{B 5}$$

We then write $$2\delta = \epsilon^4 D = \epsilon^4\alpha^4 D_0,$$

and the linear stability equations determining (U_0, V_0, W_0) and P_0 are found to be:

from continuity $$V_{0\tilde{y}}+\mathrm{i}k_0 W_0 = 0, \tag{B 6a}$$

from (X-momentum) $\cos\phi$ + (Z-momentum) $\sin\phi$,

$$U_{0\overline{T}}+V_0(\overline{u}_{01}\cos\phi+\overline{w}_{01}\sin\phi)_{\tilde{Y}} = \{\partial_{\tilde{Y}^2}-k_0^2\}\,U_0, \tag{B 6b}$$

from the Y-momentum,

$$V_{0\overline{T}}-D_0\overline{u}_{01}U_0\cos\phi = -P_{0\tilde{Y}}+\{\partial_{\tilde{Y}^2}-k_0^2\}\,V_0, \tag{B 6c}$$

from (Z-momentum) $\cos\phi$ − (X-momentum) $\sin\phi$,

$$W_{0\overline{T}} = -\mathrm{i}k_0 P_0+\{\partial_{\tilde{Y}^2}-k_0^2\}\,W_0,$$

which must be solved subject to

$$U_0 = V_0 = W_0 = 0, \quad \tilde{Y} = 0, \quad \tilde{Y}\to\infty. \tag{B 7}$$

These equations again correspond to (2.10) of Hall (1984) with (k, T) replaced by $((2/\Omega)^{\frac{1}{2}}k_0, (2/\Omega)^{\frac{1}{2}}(|p_{01}|^2/\Omega^3)\,D_0)$. Thus at the inner bend instability occurs for

$$D_0|p_{01}|^2 > 8.48\Omega^{+\frac{7}{2}}, \tag{B 8}$$

with $$\Omega = \frac{(1+B^2)}{60}.$$

We can use (B 8) to find out whether, for a given frequency, an oblique wave is unstable at a smaller amplitude than is a two-dimensional one. Suppose that A denotes a typical velocity in the ξ direction of the Tollmien–Schlichting wave. It follows from (B 4) and the definitions of u_{01}, etc. that without any loss of generality we can take

$$A = |p_{01}|\alpha^{\frac{1}{4}}\Omega^{-1}(1+B^2)^{\frac{1}{2}}.$$

The condition (B 8) then can be written in the form

$$DA^2 > 8.48\omega^{\frac{3}{2}}\{1+B^2\}, \tag{B 9}$$

where the Dean number D, frequency Ω and amplitude A are independent of B which determines the orientation of the wave. It follows from (B 9) that instability occurs first for $B = 0$ which corresponds to a two-dimensional Tollmien–Schlichting wave. We conclude that the two-dimensional form of the Tollmien–Schlichting wave is the most unstable to Taylor–Görtler vortices.

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